



Spectral representation theory and stability of the multiplicative Dhombres functional equation in f -algebras

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Abstract. We describe a method of extending certain stability results valid for real-valued functions to the class of functions with range in an f -algebra. The method is based on the Spectral Representation Theory for Riesz spaces. Details will be presented for the multiplicative Dhombres functional equation

$$(F(x) + F(y))(F(x + y) - F(x) - F(y)) = 0.$$

In this note we use the Ogasawara–Maeda Spectral Representation Theorem for Riesz spaces which will be firstly adapted to the f -algebras reality.

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1. Introduction

In this paper we investigate the possibility of applying the Spectral Representation Theory for Riesz spaces (SRT) with the purpose to develop a method of extending some results in the stability theory of functional equations for real-valued functions to the class of functions taking values in f -algebras. Some recent results in a similar direction can be found in [3, 4] (see also [5, 6]). The SRT for Riesz spaces provides us with a representation of vectors of a given Riesz space L by extended (admitting infinite values) real continuous functions on a certain topological space X which are finite on a dense subset of X ; we denote this class of functions by $C^\infty(X)$. In other words a Riesz space L , under some additional assumptions, is Riesz isomorphic to a Riesz subspace of $C^\infty(X)$. Unfortunately it appears that, in general, $C^\infty(X)$ is not necessarily a Riesz space.

As we are going to work in a reality of f -algebras, first of all we shall discuss the compatibility of the Riesz representation isomorphism (isomorphism between a given Riesz space and the space of its representatives) with the structure of an f -algebra. In the following we assume that an f -algebra L is Archimedean and possesses a multiplicative identity. It appears that then the multiplicative identity is a weak order unit in L . Thus we can make use of the Ogasawara–Maeda Spectral Representation Theorem (OMSRT) for Riesz spaces with a weak order unit which ensures the existence of a Riesz isomorphism between L (treated as a Riesz space) and a certain Riesz subspace of extended real continuous functions $C^\infty(X)$. Moreover, X appears to be extremally disconnected and therefore $C^\infty(X)$, with respect to pointwise addition, pointwise scalar multiplication and pointwise order, is a Riesz space. Furthermore, any real continuous function on a dense subset of an extremally disconnected space X can be uniquely extended to an extended real continuous function on the whole space X . This enables us to define a multiplication of extended real continuous functions in such a way that it makes $C^\infty(X)$ into an f -algebra. According to this the Riesz representation isomorphism, whose existence follows from the OMSRT, maps an f -algebra with a multiplicative identity into another f -algebra with a multiplicative identity. Next, by the OMSRT, an image of the multiplicative identity e in L is the multiplicative identity $\hat{e} \equiv 1$ in $C^\infty(X)$. Therefore, the Riesz representation isomorphism, as a positive operator, has to be multiplicative. Summarizing all the above, any Archimedean f -algebra with a multiplicative identity is f -algebra isomorphic to an f -subalgebra of the f -algebra $C^\infty(X)$ (see Sects. 3 and 4).

We are going to show how the above theory can be applied to the Ulam–Hyers stability theory. By way of example we consider the Dhombres multiplicative functional equation

$$(F(x) + F(y))(F(x + y) - F(x) - F(y)) = 0. \quad (1)$$

This equation is usually referred to as a conditional Cauchy equation with the condition dependent on the unknown function, since it can be written in the evident conditional form

$$F(x) + F(y) \neq 0 \implies F(x + y) = F(x) + F(y). \quad (2)$$

Clearly, Eqs. (1) and (2) are equivalent if there are no divisors of zero in the target space of F . For the detailed description of the solution of (2) we refer the reader to the paper of Dhombres and Ger [7].

An affirmative answer to the question of the Ulam–Hyers stability of equation (2), in the class of functions mapping an Abelian group into a Banach space, was given in [2] and, in a more general setting, in [1]. The stability of the Dhombres equation in the conditional form (2) in the class of functions mapping an Abelian group into a Riesz space was investigated in [4] with the use

of the Spectral Representation Theory for Riesz spaces, namely the Johnson–Kist Representation Theorem. On the other hand, in [2] it is proved that in the class of complex functions defined on an Abelian group equation (1) is superstable in the sense of Baker. For the readers' convenience we quote this result here, as we are going to make use of it in the sequel.

Theorem 1 (cf. [2, Theorem 2]). *Let $(G, +)$ be an Abelian group. If, for some $\varepsilon \geq 0$, a function $f : G \rightarrow \mathbb{K}$, ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) satisfies*

$$|(f(x) + f(y))(f(x + y) - f(x) - f(y))| \leq \varepsilon \quad \text{for } x, y \in G, \quad (3)$$

then f is either additive or bounded with $|f(x)| \leq \sqrt{\frac{1}{2}\varepsilon}$.

In [10] Moszner proved the superstability of the multiplicative Dhombres equation in the class of functions mapping a groupoid into a finite-dimensional normed algebra without zero divisors (cf. [10, Theorem 2.5.2]).

It raises the natural question if a similar result holds true in a more general order setting. One can rewrite all the sentences of Theorem 1 for functions mapping an Abelian group G into an f -algebra L with the common meaning of the absolute value of an element $x \in L$ stemming from the order structure of L , namely $|x| = \sup\{x, -x\}$. The main goal of Sect. 5 is to prove that such an analogue of Theorem 1 does not hold, which means that the Dhombres equation (1) in f -algebras is not superstable in the sense of Baker. However, we prove that Eq. (1) is stable in the Ulam–Hyers sense, i.e. any given $f : G \rightarrow L$ satisfying inequality (3) can be approximated by a unique additive function $a : G \rightarrow L$ in the sense that the set $\{|f(x) - a(x)| : x \in G\}$ is bounded in L .

2. Preliminaries

Throughout the paper $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and \mathbb{R}_+ are used to denote the sets of all positive integers, integers, real numbers and nonnegative real numbers, respectively.

For the readers' convenience we quote basic definitions and properties concerning Riesz spaces (see [9]).

Definition 1 (cf. [9, Definition 11.1, Definition 22.1]). We say that a real linear space L , endowed with a partial order $\leq \subset L^2$, is a *Riesz space* if $\sup\{x, y\}$ exists for all $x, y \in L$ and

$$ax + y \leq az + y \quad x, y, z \in L, x \leq z, a \in \mathbb{R}_+;$$

we define the absolute value of $x \in L$ by the formula $|x| := \sup\{x, -x\} \geq 0$.

A Riesz space L is called *Archimedean* if, for each $x \in L$, the inequality $x \leq 0$ holds whenever the set $\{nx : n \in \mathbb{N}\}$ is bounded from above.

We say that L is a *Riesz algebra* if L is a Riesz space equipped with the common algebra multiplication satisfying $xy \geq 0$ whenever $x, y \geq 0$. A Riesz

algebra L is termed an f -algebra, whenever $\inf\{x, y\} = 0$ implies $\inf\{xz, y\} = \inf\{zx, y\} = 0$ for every $z \geq 0$.

There are several types of convergence that may be defined according to the order structure. One of them is *relatively uniform convergence*.

Definition 2 (cf. [9, Definition 39.1]). Let L be a Riesz space and let $u \in L_+ := \{u \in L : u \geq 0\}$. A sequence $\{f_n\}_{n \in \mathbb{N}}$ in L is said to *converge u -uniformly* to an element $f \in L$ whenever, for every $\varepsilon > 0$, there exists a positive integer n_0 such that $|f - f_n| \leq \varepsilon u$ holds for all $n \geq n_0$. A sequence $\{f_n\}_{n \in \mathbb{N}}$ in L is called a *u -uniform Cauchy sequence* whenever, for every $\varepsilon > 0$, there exists a positive integer n_1 such that $|f_m - f_n| \leq \varepsilon u$ holds for all $m, n \geq n_1$.

Let us point out that if a Riesz space L is Archimedean, the u -uniform limit of a sequence in it, if it exists, is unique. In this case the fact that $\{f_n\}_{n \in \mathbb{N}}$ converges u -uniformly to f will be denoted by $\lim_{n \rightarrow \infty}^u f_n = f$.

Definition 3 (cf. [9, Definition 39.3]). A Riesz space L is called *u -uniformly complete* (with a given $u \in L_+$) whenever every u -uniform Cauchy sequence has a u -uniform limit.

There is a large class of spaces satisfying the above conditions. In particular every Dedekind σ -complete space, such that any of its non-empty at most countable subset which is bounded from above has a supremum, is an Archimedean u -uniformly complete space for every $u \geq 0$.

Definition 4 (cf. [9, Definition 21.4], [8, 353L]). The element $e \in L_+$ is called a *strong unit* if for every $l \in L$ there exists $\alpha \in \mathbb{R}$ such that $|l| \leq \alpha e$. The element $e \in L_+$ is called a *weak unit* if the band generated by e is the whole of L . If L is Archimedean then $e \in L_+$ is a weak unit iff $\{e\}^\perp = \{0\}$, where $\{e\}^\perp$ stands for the disjoint complement of e .

3. f -algebra $C^\infty(X)$ for extremally disconnected X

In this section we define the family $C^\infty(X)$ of extended (admitting infinite values) real continuous functions on a given topological space X that are finite-valued on a dense subset of X . In general $C^\infty(X)$ is not a linear space as it is not necessarily closed with respect to addition. However, if X is extremally disconnected, i.e. the closure of every open set is open, then $C^\infty(X)$, with a suitably defined addition, pointwise scalar multiplication and pointwise order, is a Riesz space. The above mentioned facts can be found in [9, Ch. 7, §44 and §50]. Eventually we prove that an appropriate definition of multiplication makes $C^\infty(X)$ into an f -algebra.

Let us proceed with the details. Given a topological space X , any continuous mapping f of X into $R^\infty := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ with the usual topology, such that the set

$$R(f) = \{x : |f(x)| < \infty\}$$

is dense in X is called an *extended (real-valued) continuous function* on X . The set of all extended (real-valued) continuous functions on X will be denoted by $C^\infty(X)$. We consider pointwise order in $C^\infty(X)$ and pointwise multiplication by scalars. Thus, given $f, g \in C^\infty(X)$ and the finite real number α , the functions $\inf\{f, g\}$, $\sup\{f, g\}$ and αf , all members of $C^\infty(X)$ are given by

$$\begin{aligned}(\inf\{f, g\})(x) &= \inf\{f(x), g(x)\}, \\(\sup\{f, g\})(x) &= \sup\{f(x), g(x)\}, \\(\alpha f)(x) &= \alpha f(x)\end{aligned}$$

for all $x \in X$, where it is understood that $0 \cdot \infty = 0$. For any $f \in C^\infty(X)$, the set $R(f)$ is open and dense. If $f, g, h \in C^\infty(X)$ and $h(x) = f(x) + g(x)$ holds for all $x \in R(f) \cap R(g)$, then (by definition) h is called *the sum of f and g* (notation $h = f + g$). Since the set $R(f) \cap R(g)$ is dense, the function $h = f + g$ is uniquely defined if it exists. It appears that $C^\infty(X)$ is not necessarily closed with respect to the operation of addition (cf. [9, p. 295]).

Definition 5. Any subset of $C^\infty(X)$ which is closed under the operation of addition, multiplication by scalars and the taking of finite infima and suprema, is obviously a Riesz space with respect to pointwise ordering. Accordingly, any subset of this kind is called a *Riesz space of extended real continuous functions* on X .

In the following we assume that X is *extremally disconnected*, i.e. the closure of every open set is open. We will argue then, that $C^\infty(X)$ is a Riesz space and even an f -algebra with multiplicative identity under the appropriate definitions.

Theorem 2 (cf. [9, Theorem 47.1]). *Let f be a finite-valued real continuous function defined on the open subset O of the extremally disconnected space X . Then f can be uniquely extended to an extended real continuous function \bar{f} on the closure \bar{O} of O .*

This guarantees that no problem with the addition of elements of $C^\infty(X)$ can occur. Indeed, if $f, g, h \in C^\infty(X)$ and $h(x) = f(x) + g(x)$ holds for all $x \in R(f) \cap R(g)$, then such h has a unique extension in $C^\infty(X)$, since the set $R(f) \cap R(g)$ on which h is finite-valued and continuous is open and dense. The argument for scalar multiplication, as well as for lattice operations, is similar. Therefore we have

Theorem 3 (cf. [9, Theorem 47.2]). *If X is an extremally disconnected topological space then $C^\infty(X)$ is a Riesz space.*

Given $f, g \in C^\infty(X)$, the set $R(f) \cap R(g)$ is open and dense. Thus the function equal to $f(x)g(x)$ for every $x \in R(f) \cap R(g)$ is finite-valued and continuous on $R(f) \cap R(g)$. Hence, by Theorem 2, this function can be uniquely

extended to an extended continuous function on X , and let by definition this extended function be fg .

Proposition 1. *If X is an extremally disconnected topological space then $C^\infty(X)$ is an f -algebra with a multiplicative identity.*

Proof. By Theorem 3 $C^\infty(X)$ is a Riesz space. According to Theorem 2 it is easy to observe that the above defined multiplication is a ring multiplication, hence $C^\infty(X)$ is an algebra. Moreover we have $fg \geq 0$ whenever $f, g \geq 0$ and $\inf\{fh, g\} = \inf\{hf, g\} = 0$ for every $h \geq 0$, provided that $\inf\{f, g\} = 0$. The multiplicative identity in $C^\infty(X)$ is the function $e \equiv 1$. \square

4. The Ogasawara–Maeda Spectral Representation Theorem and its extension to f -algebras

In the proof of Theorem 6 we are going to make use of the Ogasawara–Maeda Spectral Representation Theorem (OMSRT) which says that an Archimedean Riesz space with a weak unit is Riesz isomorphic to some Riesz subspace of the Riesz space $C^\infty(X)$. Since we work with the reality of f -algebras rather than Riesz spaces, as multiplication is involved in the equation we deal with, at first we need to extend the classical OMSRT to the case of f -algebras. Thus, the main aim of this section is to show that an Archimedean f -algebra with a multiplicative identity is f -algebra isomorphic to the subalgebra of the f -algebra $C^\infty(X)$ defined in Sect. 3.

Theorem 4 (OMSRT) (cf. [9, Theorem 50.1]). *Let L be an Archimedean Riesz space with a weak unit $e \in L_+$. Then there exists a Riesz space \hat{L} of extended real continuous functions and a Riesz isomorphism of L onto \hat{L} . Moreover $\hat{e} \equiv 1$.*

We omit the formal details concerning the construction of the topological space on which the Ogasawara–Maeda representatives are defined as well as the definition of representatives because they exceed the scope of the paper. However, the following two observations will be useful.

Remark 1. Directly from the construction of the Ogasawara–Maeda representatives one can deduce that the Ogasawara–Maeda representative of a weak order unit $e \in L_+$ is a constant function $\hat{e} \equiv 1$.

Remark 2. The topological space on which the Ogasawara–Maeda representatives are defined appears to be extremally disconnected. For more detailed information we refer the interested reader to [9, p. 329].

In the following we will not distinguish $l \in L$ and its Ogasawara–Maeda representative if no confusion can occur.

The following theorem provides conditions under which a positive linear operator between two f -algebras is multiplicative.

Theorem 5 ([8, Proposition 353P]). *If L is an Archimedean f -algebra with multiplicative identity e then e is a weak order unit in L . Moreover, if V is another Archimedean f -algebra with multiplicative identity e' , and $T : L \rightarrow V$ is a positive linear operator such that $T(e) = e'$, then T is a Riesz homomorphism iff $T(u \cdot v) = T(u) \cdot T(v)$ for all $u, v \in L$.*

We conclude this section with the following

Proposition 2. *Let L be an Archimedean f -algebra with a multiplicative identity $e \in L_+$. Then there exist a topological space X and a Riesz subalgebra \hat{L} of the f -algebra $C^\infty(X)$ and an f -algebra isomorphism of L onto \hat{L} .*

Proof. According to Theorem 5 the multiplicative unit e in L is a weak order unit. Thus we can apply the OMSRT in order to learn that L , as a Riesz space, is Riesz isomorphic to a Riesz subspace of the Riesz space $C^\infty(X)$. By Proposition 1, according to Remark 2, $C^\infty(X)$ is an f -algebra. Moreover the Ogasawara–Maeda image of the multiplicative identity e is $\hat{e} \equiv 1$ —the multiplicative identity in the f -algebra $C^\infty(X)$. Since any Riesz homomorphism is positive, as a straightforward consequence of Theorem 5 we get that the above mentioned Riesz isomorphism is multiplicative. \square

5. Stability results

Our main stability result reads as follows.

Theorem 6. *Let $(G, +)$ be an Abelian group and let L be an Archimedean f -algebra with a multiplicative identity $e \in L_+$. Assume that L is u -uniformly complete for given $u \in L_+$. If a function $F : G \rightarrow L$ satisfies*

$$|(F(x) + F(y))(F(x + y) - F(x) - F(y))| \leq 2u^2 \quad \text{for } x, y \in G, \quad (4)$$

then there exists a unique additive function $A : G \rightarrow L$ such that

$$|F(x) - A(x)| \leq u \quad \text{for } x \in G. \quad (5)$$

The detailed proof of Theorem 6 is postponed to the next section.

Roughly speaking Theorem 6 states that the Dhombres equation in the multiplicative form (1) in f -algebras is stable in the Ulam–Hyers sense. According to Theorem 1 one may expect Eq. (1) to be superstable in the sense of Baker. But it appears that this is not the case—the Dhombres equation (1) in f -algebras fails to be superstable. To be more precise, there is a group G and an f -algebra L , satisfying all the assertions of Theorem 4, and a function $F : G \rightarrow L$ which satisfies (4) and, at the same time, is neither additive nor bounded.

Example 1. Let $B[-1, 1]$ be the Archimedean f -algebra of all bounded real functions on the interval $[-1, 1]$ with a strong unit $e \equiv 1$ with pointwise order, pointwise addition and multiplication. Then $B[-1, 1]$ is e -uniformly complete. Let $u \in B[-1, 1]$ be given by

$$u(s) := \begin{cases} |s| & \text{if } s \in [-1, 0] \\ 0 & \text{if } s \in (0, 1]. \end{cases}$$

Let us define $F : \mathbb{R} \rightarrow B[-1, 1]$ by

$$F(x)(s) := \begin{cases} s & \text{if } s \in [-1, 0] \\ sx & \text{if } s \in (0, 1] \end{cases} \quad \text{for } x \in \mathbb{R},$$

which is, clearly, neither bounded nor additive. Moreover F and u satisfy (4).

Remark 3. The constant of approximation in inequality (4) in Theorem 6 is the best possible one. In order to observe that it suffices to examine our Example 1 once again. An additive function $A : \mathbb{R} \rightarrow B[-1, 1]$ of the form

$$A(x)(s) := \begin{cases} 0 & \text{if } s \in [-1, 0] \\ sx & \text{if } s \in (0, 1] \end{cases} \quad \text{for } x \in \mathbb{R}$$

approximates F with $|F(x) - A(x)| = u$ for $x \in \mathbb{R}$.

6. Auxiliary results and proofs

As a straightforward consequence of Theorem 1 we have the following

Corollary 1. Let $(G, +)$ be an Abelian group. If for some $\varepsilon \geq 0$ a function $f : G \rightarrow \mathbb{K}$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) satisfies

$$|(f(x) + f(y))(f(x + y) - f(x) - f(y))| \leq \varepsilon \quad \text{for } x, y \in G,$$

then there exists a unique additive function $a : G \rightarrow \mathbb{R}$ such that

$$|f(x) - a(x)| \leq \sqrt{\frac{1}{2}\varepsilon} \quad \text{for } x, y \in G.$$

Proof. If f is unbounded, then we define a to be equal to f on G . In the opposite case we put $a(x) = 0$ for $x \in G$. \square

In the sequel the following technical Lemma will be used.

Lemma 1. Let $(G, +)$ be an Abelian group and let $f : G \rightarrow \mathbb{R}$. If there exists an additive function $a : G \rightarrow \mathbb{R}$ and a real number $\alpha \geq 0$ such that

$$|f(x) - a(x)| \leq \alpha, \tag{6}$$

for all $x \in G$, then for $f_n(x) := \frac{1}{n}f(nx)$ ($n \in \mathbb{N}, x \in G$) we have

$$|f_n(x+y) - f_n(x) - f_n(y)| \leq \frac{3}{n}\alpha, \quad (7)$$

$$|f(x) - f_n(x)| \leq \left(1 + \frac{1}{n}\right)\alpha, \quad (8)$$

$$|f_n(x) - f_k(x)| \leq \left(\frac{1}{n} + \frac{1}{k}\right)\alpha \quad (9)$$

for all $x, y \in G$ and $n, k \in \mathbb{N}$.

Proof. Routine. □

Proof of Theorem 6. The main idea of the proof is based on the use of Proposition 2 which, after avoiding some difficulties concerning infinite values, enables us to reduce the problem to the real case. Then we are able to apply Corollary 1. Eventually we come back to the original f -algebra L .

The proof runs in three steps.

Step 1. We will prove that the sequence $F_n(x) = \frac{1}{n}F(nx)$ ($n \in \mathbb{N}$) is u -uniformly convergent for $x \in G$ and we define $A : G \rightarrow L$ as the u -uniform limit of F_n . Consider $F : G \rightarrow L$ satisfying (4). According to Proposition 2 $F(x), F(y), F(x+y), 2u^2 \in C^\infty(X)$. Let us define

$$W_x := R(F(x)) \cap R(F(0)) \cap R(u). \quad (10)$$

Replacing x by $-x$ and y by x in (4) we infer that $W_x \subset R(F(-x))$. Now, suppose that $W_x \subset R(F(nx))$ for given $n \in \mathbb{N}$ and apply (4) with x and y replaced by $(n+1)x$ and $-x$, respectively, in order to obtain $W_x \subset R(F((n+1)x))$. By induction we get

$$W_x \subset R(F(nx)) \quad \text{for } n \in \mathbb{N}. \quad (11)$$

On the other hand, using (4) with nx in place of x and $-nx$ in place of y , we get $W_x \cap R(F(nx)) \subset R(F(-nx))$. This along with (11) yields

$$W_x \subset R(F(z)) \quad \text{for } z \in \langle x \rangle, \quad (12)$$

where $\langle x \rangle := \{kx : k \in \mathbb{Z}\}$.

Let us fix an arbitrary $x \in G$ and rewrite (4) for $z, w \in \langle x \rangle$ as

$$|(F(z) + F(w))(F(z+w) - F(z) - F(w))| \leq 2u^2.$$

According to Proposition 2 $F(z), F(w), F(z+w), 2u^2 \in C^\infty(X)$. Thus, for arbitrarily fixed $s \in W_x$ we have

$$|(F(z)(s) + F(w)(s))(F(z+w)(s) - F(z)(s) - F(w)(s))| \leq 2u^2(s)$$

for $z, w \in \langle x \rangle$, which means that $\langle x \rangle$ and $F(\cdot)(s)|_{\langle x \rangle}$ satisfy all the assumptions of Corollary 1. By Corollary 1 and Lemma 1, for any $n, k \in \mathbb{N}$ we have

$$|F_n(x)(s) - F_k(x)(s)| \leq \left(\frac{1}{n} + \frac{1}{k}\right) u(s)$$

and

$$|F(x)(s) - F_n(x)(s)| \leq \left(1 + \frac{1}{n}\right) u(s).$$

Since $s \in W_x$ is arbitrary, W_x is open and dense, moreover all the functions in the above inequalities are continuous, we obtain

$$|F_n(x) - F_k(x)| \leq \left(\frac{1}{n} + \frac{1}{k}\right) u \quad \text{for } n, k \in \mathbb{N} \quad (13)$$

and

$$|F(x) - F_n(x)| \leq \left(1 + \frac{1}{n}\right) u \quad \text{for } n \in \mathbb{N}. \quad (14)$$

Inequality (13) means that $(F_n(x))_{n \in \mathbb{N}}$ is a u -uniform Cauchy sequence in a u -uniformly complete Riesz space and, therefore, it is relatively uniformly convergent. This, according to the fact that $x \in G$ was arbitrarily fixed, proves that $A : G \rightarrow L$ given by

$$A(x) := \lim_{n \rightarrow \infty} {}^u F_n(x) \quad \text{for } x \in G \quad (15)$$

is well-defined.

Letting $n \rightarrow \infty$ in (14) we obtain (5).

Step 2. We shall prove the additivity of A . Let, for arbitrarily fixed $x, y \in G$ the sets W_x and W_y be defined as in (10) and let $W_{x,y} := W_x \cap W_y$. We would like to show that

$$W_{x,y} \subset R(F(z)) \quad \text{for } z \in \langle x, y \rangle. \quad (16)$$

Let us consider $z \in \langle x \rangle$ and $w \in \langle y \rangle$. By (12) $W_{x,y} \subset W_x \subset R(F(z))$ and $W_{x,y} \subset W_y \subset R(F(w))$. Using (4) with x and y replaced by $z + w$ and $-w$, respectively, we get

$$|(F(z + w) + F(-w))(F(z) - F(z + w) - F(-w))| \leq 2u^2$$

which enables us to observe that $W_{x,y} \subset R(F(z + w))$. This completes the proof of (16).

By (4) applied to $z, w \in \langle x, y \rangle$ we have

$$|(F(z) + F(w))(F(z + w) - F(z) - F(w))| \leq 2u^2.$$

According to Proposition 2 $F(z), F(w), F(z+w), 2u^2 \in C^\infty(X)$ which results in

$$|(F(z)(s) + F(w)(s))(F(z+w)(s) - F(z)(s) - F(w)(s))| \leq 2u^2(s)$$

for $z, w \in \langle x, y \rangle$, with arbitrarily fixed $s \in W_{x,y}$. This means that $\langle x, y \rangle$ and $F(\cdot)(s)|_{\langle x, y \rangle}$ satisfy all the assumptions of Corollary 1. By Corollary 1 and Lemma 1 we have

$$|F_n(x+y)(s) - F_n(x)(s) - F_n(y)(s)| \leq \frac{3}{n}u(s) \quad \text{for } n \in \mathbb{N}.$$

Since the last inequality is valid for any s from the open and dense set $W_{x,y}$ and all the functions in the above inequality are continuous, we obtain

$$|F_n(x+y) - F_n(x) - F_n(y)| \leq \frac{3}{n}u \quad \text{for } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ and considering the definition of A we have $A(x+y) = A(x) + A(y)$ which proves the additivity of A as x and y were chosen arbitrarily.

Step 3. To prove the uniqueness of A suppose that we have two additive functions $A_1, A_2 : G \rightarrow L$ satisfying (5). Then

$$|A_1(x) - A_2(x)| \leq 2u \quad \text{for } x \in G$$

and, with nx in place of x ,

$$n|A_1(x) - A_2(x)| \leq 2u \quad \text{for } x \in G, n \in \mathbb{N}$$

which immediately yields $A_1 = A_2$ as L is Archimedean. \square

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